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1983 J. Phys. A: Math. Gen. 16 L55

(http://iopscience.iop.org/0305-4470/16/2/002)

J. Phys. A: Math. Gen. 16 (1983) L55–L60. Printed in Great Britain

LETTER TO THE EDITOR

A theory of period-doubling bifurcations in two-dimensional reversible area preserving mappings

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Received 14 September 1982

Abstract. For a class of two-dimensional reversible area preserving mappings, a theory is developed which describes how the period doubling bifurcation entails the onset of hyperbolic instability with reflection. Henon's quadratic mapping, Chirikov's standard mapping and De Vogelaere's mapping all belong to this class.

In many recent works (Feigenbaum 1978, 1979, Derrida *et al* 1979, Helleman 1980) it is demonstrated that periodic solutions of the dynamical systems of a certain class bifurcate in the period doubling sequences. In one dimension, Feigenbaum (1978, 1979) finds an infinite sequence of period doublings in which a periodic orbit goes unstable as a parameter is varied and a pair of stable new orbits of twice the period is born. Similar behaviour has been observed in two-dimensional area preserving maps (Greene *et al* 1981, Bountis 1981, Benettin *et al* 1980a, b), although a definite relationship is yet to be demonstrated between the period doubling and the instability of the period in the case of two-dimensional area preserving maps. In this work we demonstrate explicitly that in a certain class of two-dimensional area preserving maps a periodic orbit indeed bifurcates into period doubling orbits when the orbit becomes unstable as a parameter is varied.

Consider a two-dimensional mapping T of one parameter a,

$$T:\begin{cases} x_{n+1}=\xi(x_n, y_n) \\ y_{n+1}=\eta(x_n, y_n) \end{cases} \tag{1}$$

where two variable functions ξ, η depend on the parameter a. An n-period orbit may be defined as the fixed point of T^n as $P_n^*=T^nP_n^*$. We use P to designate a point $P=(x, y)$ on a two-dimensional plane. The fixed point relationship can be rewritten as $P_n(P_0=P_n^*)=P_n^*$, where $P_{n+m}(P_0)=P_n(P_m)=T^nP_m=T^{n+m}P_0$. A quantity called the 'residue' R is convenient in discussing the stability of an orbit (Greene *et al* 1981). It is given as

$$R=(2-\operatorname{Tr} M)/4 \tag{2}$$

where M is a 2×2 matrix constructed from M_i as

$$M=\prod_{i=0}^{n-1} M_i \tag{3}$$

0305-4470/83/020055+06$02.25

and M_i is in turn given as

$$M_i = M(\bar{P}_i) = \begin{pmatrix} \xi_x(\bar{P}_i), \xi_y(\bar{P}_i) \\ \eta_x(\bar{P}_i), \eta_y(\bar{P}_i) \end{pmatrix} \tag{4}$$

with $\xi_x = \partial\xi(x, y)/\partial x$ etc and $\bar{P}_i = P_i(\bar{P}_0) = T^i\bar{P}_0$, $\bar{P}_0 = P_n^*$ being the initial point of the n-period orbit. The orbit is stable for $0<R<1$ and unstable for $R<0$ and $R>1$ except for the special cases of low order resonances. The area preserving is expressed by $\det M = 1$. Also useful is the concept of reversibility (Greene *et al* 1981) which has greatly facilitated numerical studies of period doubling sequences. In many cases of reversible maps, the period doubling occurs on a single line in the xy plane called the 'symmetry line'. The initial point $\bar{P}_0$ of a $2n$-periodic orbit (i.e. $\bar{P}_0 = P_{2n}^*$) can be found by the condition that $P_n(\bar{P}_0)$ as well as $\bar{P}_0$ itself lies on the symmetry line. It is suggestive that the bifurcation condition may be constructed on the periodic points on the symmetry line. Let T be reversible so that a 'symmetry' S exists. That is, $T = (TS)S$ with $S^2 = 1$ and $(TS)^2 = 1$, so that the inverse of T, $T^{-1} = STS$, exists. The symmetry lines are defined as the fixed line either of S or of (TS). That is, $P' = SP = P$ or $P'' = (TS)P = P$. It should be noted that these equations furnish lines instead of points, hence the name fixed line or symmetry line. Let

$$x = h(y) \tag{5}$$

represent one of the symmetry lines. The initial $\bar{P}_0 = (\bar{x}_0, \bar{y}_0)$ of a $2n$-period orbit can be found by the solution of simultaneous equations $\bar{x}_0 = h(\bar{y}_0)$ and $x_n(\bar{P}_0) = h(y_n(\bar{P}_0))$. If we define $g_n(z) = y_n(P_0)$ with $x_0 = h(z)$ and $y_0 = z$, then some roots of the equation

$$\varphi_n(z) \equiv g_n(z) - z = 0 \tag{6}$$

will give the initial point of the n-period orbit on the symmetry line. An additional condition $\bar{x}_0 = x_n(\bar{P}_0)$ for $\bar{P}_0$ whose y component is the root of equation (6) needs to be satisfied in order that $\bar{P}_0$ is the n-period initial point. We construct a function $\psi_{2n}(z)$ by

$$\psi_{2n}(z) = \varphi_n(\varphi_n(z) + z) + \varphi_n(z). \tag{7}$$

Some of the roots of the equation $\psi_{2n}(z) = 0$ then give the initial point of the $2n$ period. This observation follows from the following considerations. The T^n image of a point P_0 on the symmetry line $P_n(P_0) = (x_n(h(y_0), y_0), y_n(h(y_0), y_0))$ may or may not lie on the symmetry line. If it does not lie on the symmetry line, shift the point $P_n(P_0)$ parallel to the x axis until it intersects the symmetry line. We designate this point as (figure 1) $P'_n = (h(y_n(P_0)), y_n(P_0))$. Consider now P'_{2n}, thus obtained from $T^nP'_n$. The y component of P'_{2n}, y'_{2n} will have the same functional dependence of y_n on y_0, namely $y_n = g_n(y_0)$. Thus $y'_{2n}(y_n) = g_n(y_n)$ so that $y'_{2n}(y_0) = g_n(g_n(y_0))$. If $\bar{y}_0$ is the y component of the initial point $\bar{P}_0$ of the $2n$-period orbit, the T^n image of $\bar{P}_0$ lies on the symmetry curve. That is, $P'_n = P_n$ or $y'_{2n}(\bar{y}_0) = y_{2n}(\bar{y}_0) = \bar{y}_0$. Therefore some of the roots of

$$\psi_{2n}(z) = g_n(g_n(z)) - z \tag{8}$$

will give the y component of the initial point of the $2n$-period orbit. Substitution of equation (6) into (8) gives equation (7).

Let us suppose that the functions thus far constructed, $\varphi_n(z; a)$ and $\psi_{2n}(z; a)$, are smooth both in z and the parameter a. We assume that the period doubling bifurcation

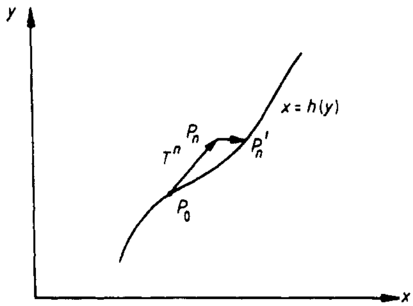

Figure 1. The point P_n is the T^n image of a point P_0 and P_n' is obtained by shifting P_n parallel to the x axis until it intersects the symmetry line $x = h(y)$.

occurs at $a = a_n$ and describe in the following what happens to these functions of φ_n and ψ_{2n} in the vicinity of $a = a_n$. Possibly two fixed points $\bar{y}_0 = y^*_{2n}(a > a_n)$ and $\bar{y}'_0 = y^{*\prime}_{2n}(a > a_n)$ are created on both sides of $\bar{y}_0 = y^*_n(a > a_n)$ as a passes $a = a_n$ (figure 2). We let $y_1 = \min\{y^*_{2n}(a > a_n), y^{*\prime}_{2n}(a > a_n)\} - d$ and $y_2 = \max\{y^*_{2n}(a > a_n), y^{*\prime}_{2n}(a > a_n)\} + d$ with $d > 0$ and suppose that $\psi_{2n}(y_1) > 0$; $\psi_{2n}(y_2) < 0$. Then it is easy to see that both $\varphi_n(y, a < a_n)$ and $\psi_{2n}(y, a < a_n)$ cross the y axis at $\bar{y}_0 = y^*_n(a < a_n)$ with negative slopes, since an n-period orbit is also a $2n$-period orbit. As a increases the negative slope, $\varphi'_n(y^*_n, a < a_n)$, becomes steeper until $\varphi'_n(y^*_n, a = a_n) = -2$ while

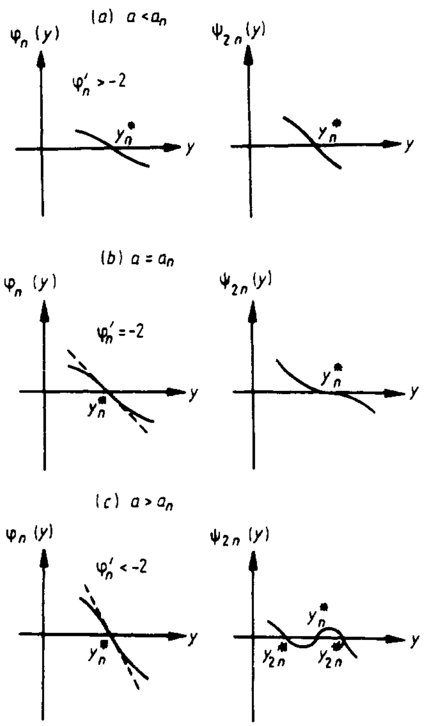

Figure 2. The behaviour of two functions φ_n and ψ_{2n} in the vicinity of $a = a_n$ where the period doubling bifurcation occurs as the parameter a changes from just below $a = a_n$ to just above. The functions φ_n and ψ_{2n} are given in equations (6) and (7).

$|\psi'_{2n}(y_n^*, a < a_n)|$ gets smaller and becomes zero at $a = a_n$, i.e. $\psi'_{2n}(y_n^*, a = a_n) = 0$. This can be seen by differentiating equation (7),

$$\psi'_{2n}(z) = \varphi'_n(\varphi_n(z) + z)(\varphi'_n(z) + 1) + \varphi'_n(z),$$

and putting $z = y_n^*$ to get

$$\psi'_{2n}(y_n^*) = \varphi'_n(y_n^*)(\varphi'_n(y_n^*) + 2). \tag{9}$$

When $\varphi'_n(y_n^*, a > a_n)$ passes below the value -2, $\psi'_{2n}(y_n^*, a > a_n)$ becomes positive and it becomes possible to have two roots of $\psi_{2n}(y) = 0$ on both sides of $y_n^*(a > a_n)$. We also show that the period doubling bifurcation entails the instability of the n-period orbit for a certain class of 2D area preserving mappings by showing that

$$\varphi'_n(y_n^*, a) = -2R. \tag{10}$$

From the observations of Greene *et al* (1981) we see then that as R increases from $R < 1$ to $R > 1$, corresponding to $a < a_n$ and $a > a_n$, the stable elliptical orbit ($0 < R < 1$) becomes unstable at $R = 1$ ($a = a_n$) and it changes into a hyperbolic orbit with reflection ($R > 1$) and thus period doubling bifurcation ensues.

In the following we prove that equation (10) is satisfied for a certain class of T, namely those constructed from a one-dimensional non-invertible map $x_{n+1} = F(x_n)$ as follows (Ott 1981):

$$T = \begin{cases} x_{n+1} = -y_n + F(x_n) \\ y_{n+1} = x_n. \end{cases} \tag{11}$$

Restricting to the class of T given by equation (11), we find two complementary symmetries (Greene *et al* 1981)

$$S = \begin{cases} x_{t+1} = y_t \\ y_{t+1} = x_t \end{cases} \tag{12}$$

and

$$TS = \begin{cases} x_{t+1} = -x_t + F(y_t) \\ y_{t+1} = y_t. \end{cases} \tag{13}$$

We now consider the periodic orbits on the symmetry line of TS, i.e.

$$x = F(y)/2. \tag{14}$$

We present two lemmas which are useful for the proof of equation (10).

Lemma 1. The x components of an even n-period orbit starting from the initial point $P_0 = (\bar{x}_0 = F(\bar{y}_0)/2, \bar{y}_0)$ satisfy

$$\bar{x}_{n/2-l} = \bar{x}_{n/2+l-2} \qquad \text{for} \quad l = 1, 2, \ldots, n/2 - 1. \tag{15}$$

Lemma 1 follows from the fact that the midpoint of the even n-period orbit, $(\bar{x}_{n/2}, \bar{y}_{n/2})$ lies on the symmetry line.

We define

$$\bar{M} = \prod_{i=0}^{n-2} M_i \tag{16}$$

where M_i's are given in equation (4).

Lemma 2. The sum of the off-diagonal elements of $\bar{M}$ is zero. That is,

$$\bar{M}_{12}+\bar{M}_{21}=0 \tag{17}$$

where

$$\bar{M}=\begin{pmatrix}\bar{M}_{11} & \bar{M}_{12}\\ \bar{M}_{21} & \bar{M}_{22}\end{pmatrix}.$$

Lemma 2 follows from the fact that M_i's are of a special form.

Making use of lemmas 1 and 2 we can prove equation (10).

The mapping given by equation (11) covers quite a large class of mappings. When we choose $F(x)=1-ax^2$, it is the Henon quadratic mapping studied by Bountis (1981), and when we make appropriate area preserving transformations, we can show that T with $F=2x-\mu \sin 2\pi x$ is equivalent to Chirikov's (1979) standard mapping, T with $F(x)=2f(x)$ to the general De Vogelaere mapping (Greene *et al* 1981) and T with $F(x)=2[px-(1-p)x^2]$, the quadratic De Vogelaere mapping studied by Greene *et al* (1981). The T's with $F=2x-\mu f(x)$ of various $f(x)$ are also equivalent to the mappings studied extensively by Benettin *et al* (1980a, b). However, it still remains to be checked whether the theory formulated is applicable to all of the above-mentioned class of mappings. The conditions are that the periodic orbits exist on the symmetry lines of equation (14) and the stability breaks down smoothly as the parameter changes. If they do, the period doubling bifurcations occur as the periodic points of the orbit become hyperbolically unstable with reflection. We are testing this behaviour for a class of nonlinear mappings and the result will be reported elsewhere.

We wish to acknowledge the support of the Ministry of Education, Republic of Korea through a grant to the Research Institute for Basic Sciences, Seoul National University.

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